Schramm-Loewner Evolution

A quick overview

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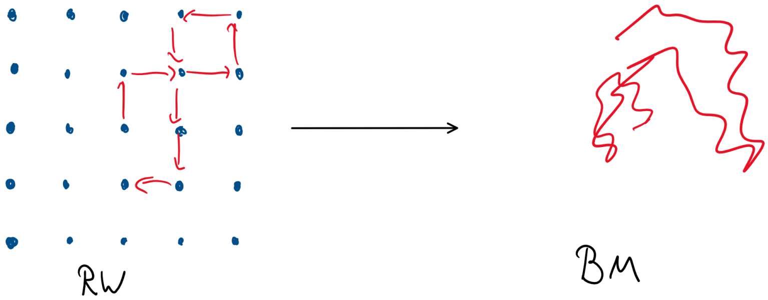
Outline

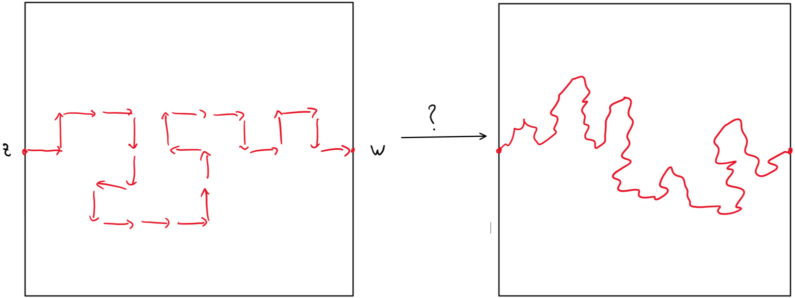
Reference:
Conformally Invariant Processes in the Plane - G. Lawler

BM from random walks



Scaling limit Brownian motion.

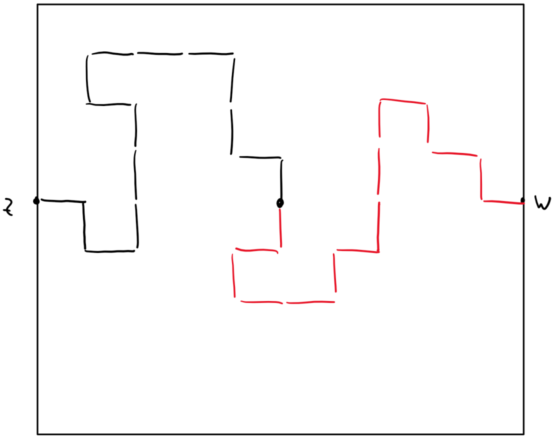
Self-avoiding random walk



Conjectured Scaling limit $SLE_{\frac{8}{3}}$

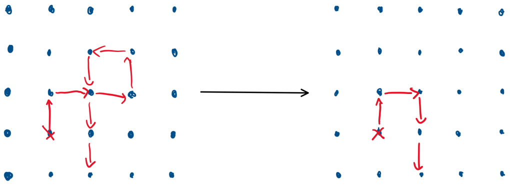
Domain Markov Property

Measure $\mu_D^{\#}(z, w)$
conditioned on initial curve segment
$\gamma(0, t]$ is equal to $\mu^{\#}_{D\setminus\gamma(0,t]}(\gamma(t), w)$



Loop-erased random walk

- (DMP)
- Conformally invariant in the limit
- Scaling limit SLE_2. (LSW ’04)



Scaling Limits?

- Erasing loops from BM? - Hard
- Conformal invariance to the rescue.
- CI + DMP $\implies$ SLE.

Definition (Conformal map)

$f : D_1 \to D_2$ is conformal if it is bijective and holomorphic.

Brownian Motion is Conformally invariant

Theorem

- *$f : D_1 \to D_2$ conformal, $0 \in D_1, D_2$, $f(0) = 0$.*
- *$W = X + iY$ planar Brownian motion*
- *Define $\tau_{D_1} := \inf\{t \geq 0 : W_t \notin D_1\}$*

Then $f(W_t)|_{[0,\tau_{D_1}]}$ is a time-changed Brownian motion.

$\int_0^{\sigma_s} |f'(B_r)|^2 dr = s$, then $f(W_{\sigma_s})$ is BM in D_2.
Locally scaling + rotation.

Brownian Motion is conformally invariant

Theorem

$\widetilde{W}_t = f(W_{\sigma_t})$ is Brownian motion.

Proof:

$f = u + iv$. By Ito's Lemma + Cauchy Riemann:

$$\begin{aligned} d(u(W_t)) &= u_x(W_t)dX_t + u_y(W_t)dY_t \\ \implies < u(W) >_t &= \int_0^t |f'(W_s)|^2 ds \end{aligned}$$

Thus $< \widetilde{X} >_t = < \widetilde{Y} >_t = t, < \widetilde{X}, \widetilde{Y} > = 0$.

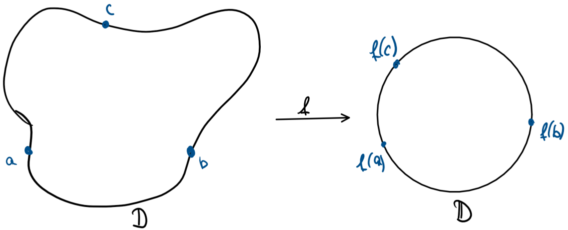
+ Lévy characterization.

Riemann Mapping Theorem

Theorem

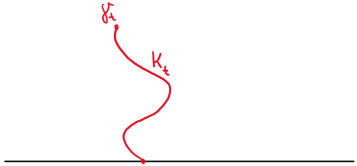
D non-empty, simply connected proper subset of $\mathbb{C}$, then there exists a conformal $f : D \to \mathbb{D}$.

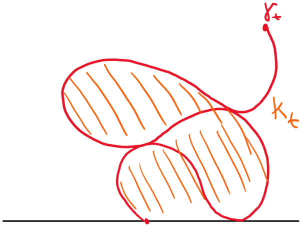
Three real degrees of freedom.



Mapping out

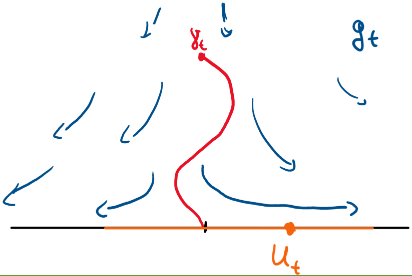
- $\gamma : [0, \infty) \to \mathbb{H}, \gamma(0) = 0, \gamma(\infty) = \infty$.
- K_t^c = unbounded component of $\mathbb{H} \setminus \gamma[0, t]$.





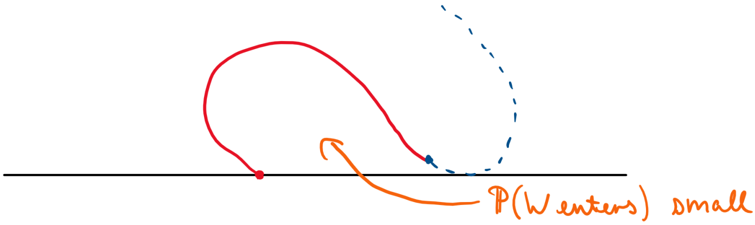
Mapping out

- $\gamma : [0, \infty) \to \mathbb{H}, \gamma(0) = 0, \gamma(\infty) = \infty$.
- K_t^c = unbounded component of $\mathbb{H} \setminus \gamma[0, t]$.
- $g_t : K_t^c \to \mathbb{H}$ conformal, $g_t(\infty) = \infty$.
- Expand at ∞: $g_t(z) = a_1 z + a_0 + a_{-1} z^{-1} + \dots$ with $a_i \in \mathbb{R}$.
- $\exists!\ g_t$ with $a_1 = 1, a_0 = 0$.
- compact $\mathbb{H}$ hull $K \leftrightarrow$ mapping out function g



Capacity

$g_t(z) = z + a_{-1}z^{-1} + \dots$
Then $\mathrm{hcap}(K_t) := a_{-1} = \lim_{y\to\infty} y\mathbb{E}^{iy}[\mathrm{Im}(W_{\tau_{K_t\cup\mathbb{R}}})]$.



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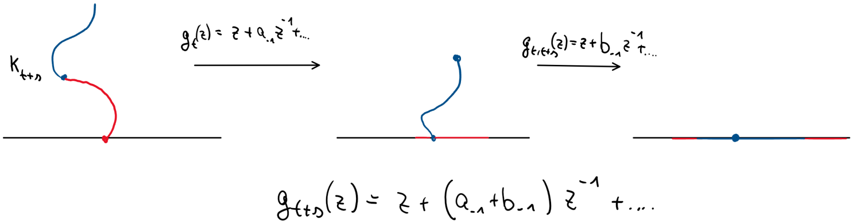
Lemma (Additivity)

$hcap(K_{t+s}) = hcap(K_t) + hcap(g_t(K_{t+s} \setminus K_t))$



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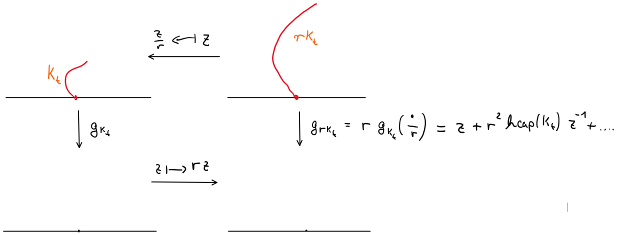
Lemma (Scaling)

$$hcap(rK_t) = r^2 hcap(K_t)$$



Loewner Evolution

Parametrize γ by hcap.

Theorem

Let γ_t be a simple curve from 0 to ∞ parametrized s.t. hcap$(K_t) = 2t$. $\tau_z := \inf\{t \geq 0 : z \in K_t\}$, $U_t = g_t(\gamma_t)$ Then

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t} \text{ for } t \in [0, \tau_z],\ g_0(z) = z.$$

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Remark: Can extend to non-simple curves γ:
Sets K_t derived from Loewner chains "continuously increasing hulls":

$$\bigcap_{\delta > 0} \overline{K_{t,t+\delta}} = U_t$$

Schramm Loewner Evolution

Study U_t instead of γ_t. Invert construction:

Definition

- W_t BM, $\kappa \geq 0$.
- Solve $\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} W_t}$, $g_0(z) = z$ for $t < \tau_z = \sup\{t \geq 0, g_t(z) \text{ well defined}\}$.
- $K_t = \{z \in \mathbb{H} : \tau_z \leq t\}$
- γ curve generating $(K_t)_{t \geq 0}$ is SLE_κ.

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- $K_t = \{z \in \mathbb{H} : \tau_z \leq t\}$
- γ curve generating $(K_t)_{t \geq 0}$ is SLE$_\kappa$.

Proposition (Rohde-Schramm, Lawler-Schramm-Werner)

γ is well defined.

Remark: SLE$_\kappa$ is really a measure $\mu^{\#}_{\mathbb{H}}(0, \infty)$ on paths going from 0 to ∞ up to time reparametrization.

Scale invariance

Proposition

If γ is SLE_κ then $\tilde{\gamma} : t \mapsto r\gamma(\frac{t}{r^2})$ is also SLE_κ.

Scale invariance

Proposition

If γ is SLE$_\kappa$ then $\tilde{\gamma} : t \mapsto r\gamma(\frac{t}{r^2})$ is also SLE$_\kappa$.

Proof: $\tilde{g}_t(z) = rg_{t/r^2}(z/r)$ is mapping out function of $\tilde{\gamma}$.

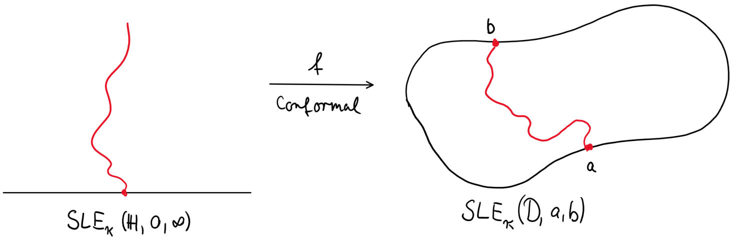
$\implies \dot{\tilde{g}}_t(z) = \frac{2}{\tilde{g}_t(z) - r\sqrt{\kappa}W(t/r^2)} \overset{d}{=} \frac{2}{\tilde{g}_t(z) - \sqrt{\kappa}W(t)}$.

SLE on other domains

D simply connected domain, $z, w \in \partial D$, $f : \mathbb{H} \to D$ conformal with $f(0) = z$, $f(\infty) = w$.

Definition

$\mu_D^{\#}(z, w)$ is the image of $\mu_{\mathbb{H}}^{\#}(0, \infty)$ under f.



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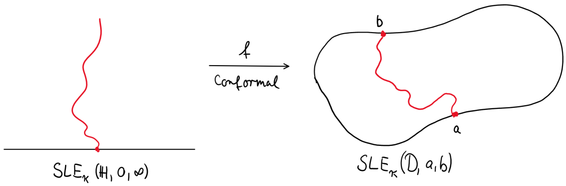
$\mu_D^{\#}(z, w)$ is the image of $\mu_{\mathbb{H}}^{\#}(0, \infty)$ under f.

f not unique - any other such map written as $f_1(z) = f(rz), r > 0$. Still well defined by scale invariance.
New mapping out function $h_t(z) = f(g_t(f^{-1}(z)))$



Conformal invariance and Domain Markov Property

- For any D, a, b as above given measures $\mu_D(a, b)$ on curves modulo time reparametrization.
- (CI): $f_*(\mu_{D_1}(a, b)) = \mu_{D_2}(z, w)$ for conformal f.
- (DMP): $\mu_D(a, b)(\cdot\ |\gamma|_{[0,\tau]}) = \mu_{D\setminus\gamma([0,\tau])}(\gamma(\tau), b)$

Theorem (Schramm '00)

$(\mu_D(a, b))$ satisfies (CI) and (DMP) $\iff$ $\mu_D(a, b) \overset{d}{=} SLE_\kappa$ for some $\kappa \geq 0$.

Conformal invariance and Domain Markov Property

Theorem (Schramm '00)

$(\mu_D(a,b))$ satisfies (CI) and (DMP) $\iff$ $\mu_D(a,b) \overset{d}{=} SLE_\kappa$ for some $\kappa \geq 0$.

Proof sketch:
Enough to consider $\mu_{\mathbb{H}}(0,\infty)$. Let U_t be the Loewner transform of K_t. Then

$$\underbrace{U_t^\lambda := \lambda U_{t/\lambda^2}}_{\text{LT of } \lambda K_t}, \quad \underbrace{U_t^{(s)} := U_{s+t} - U_s}_{\text{LT of } K_{s,t} = g_s(K_{t+s} \setminus K_s)}.$$

Thus

- K_t scale inv $\iff$ U_t scale inv.
- K_t DMP $\iff$ (U_t) stationary indep increments.

Phases of SLE

Proposition (Rohde-Schramm)

- $\kappa \in [0, 4]$: γ *is simple.*
- $\kappa \in (4, 8)$: γ *is self-intersecting with* 0 *Lebesque measure.*
- $\kappa \geq 8$: γ *is space-filling.*

$\hat{g}_t(z) = \frac{g_t(z) - \sqrt{\kappa} W_t}{\sqrt{\kappa}}$ satisfies

$$d\hat{g}_t(z) = \frac{2/\kappa}{\hat{g}_t(z)} dt - dW_t.$$

$\implies n = \frac{4}{\kappa} + 1$ dimension of Bessel process.