Dimers and M-Curves: Limit Shapes from Riemann Surfaces

BIMSA - Representation Theory, Integrable Systems and Related Topics

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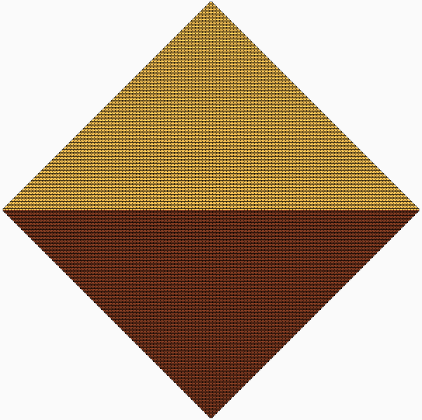
Papers

- Dimers and M-Curves [Bobenko-B-Suris, 2024]
- Dimers and M-Curves: Limit Shapes from Riemann Surfaces [Bobenko-B, 2024]

The Dimer Model

- G planar bipartite graph, dimer model: random perfect matching = dimer configuration.



Trivial Aztec diamond dimer configuration.

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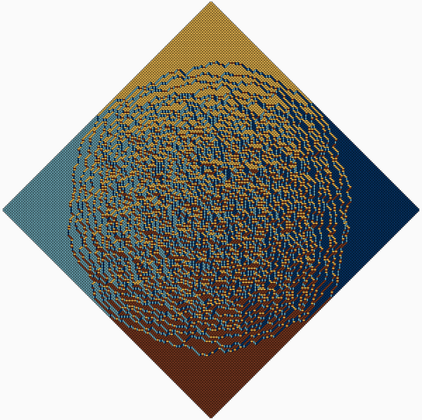

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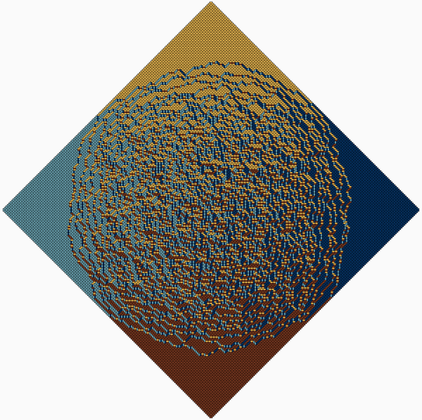

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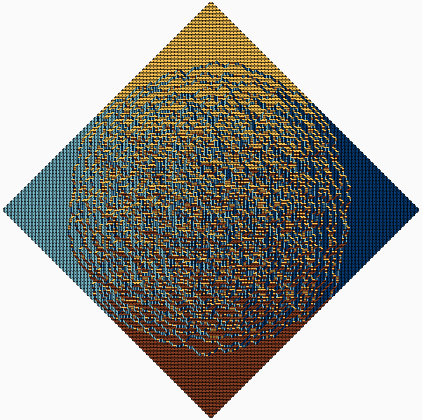
$$h \to \arg\min_f \int_\Omega \sigma(\nabla f).$$



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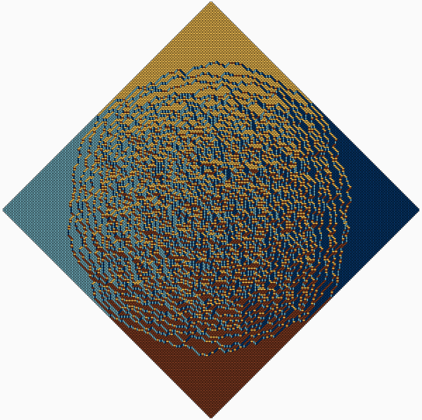
$$h \to \arg\min_{f} \int_{\Omega} \sigma(\nabla f).$$

- Euler-Lagrange equation with singularities

$$\mathrm{div}(\nabla \sigma(\nabla h)) = 0$$



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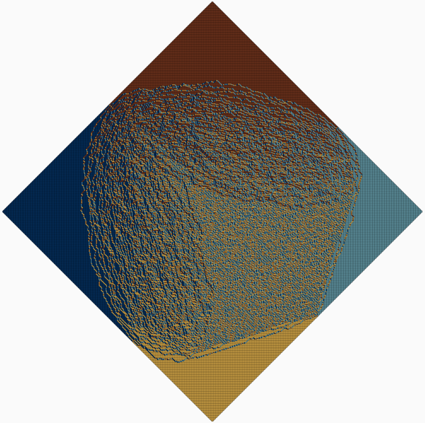
Doubly Periodic Weights

- $\mathbb{P}(D) = \frac{1}{Z} \prod_{e \in D} \nu(e)$



Weighted random Aztec diamond

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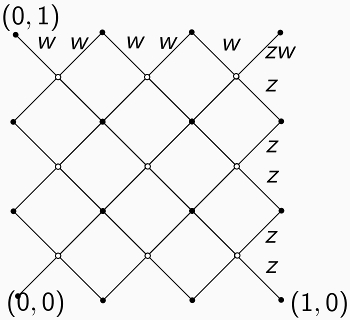
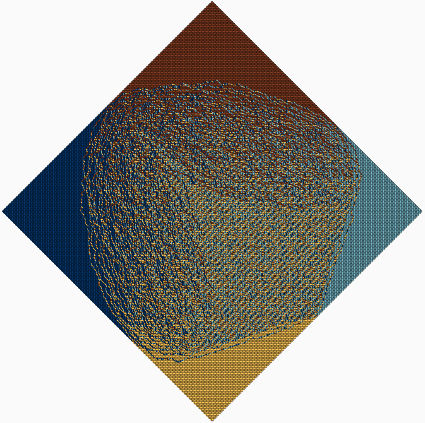
- Doubly periodic weights. Fundamental domain:





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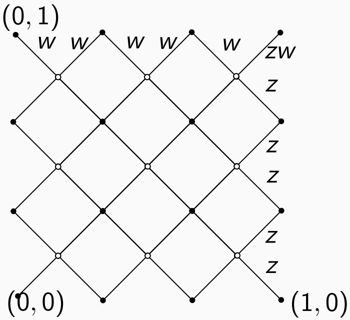
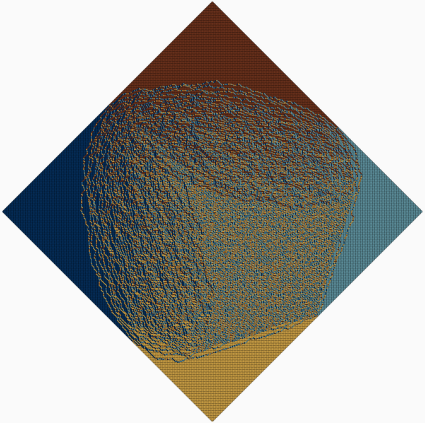


- $\det K(z, w) = 0$ spectral curve defines σ [Kenyon-Okounkov-Sheffield '07]

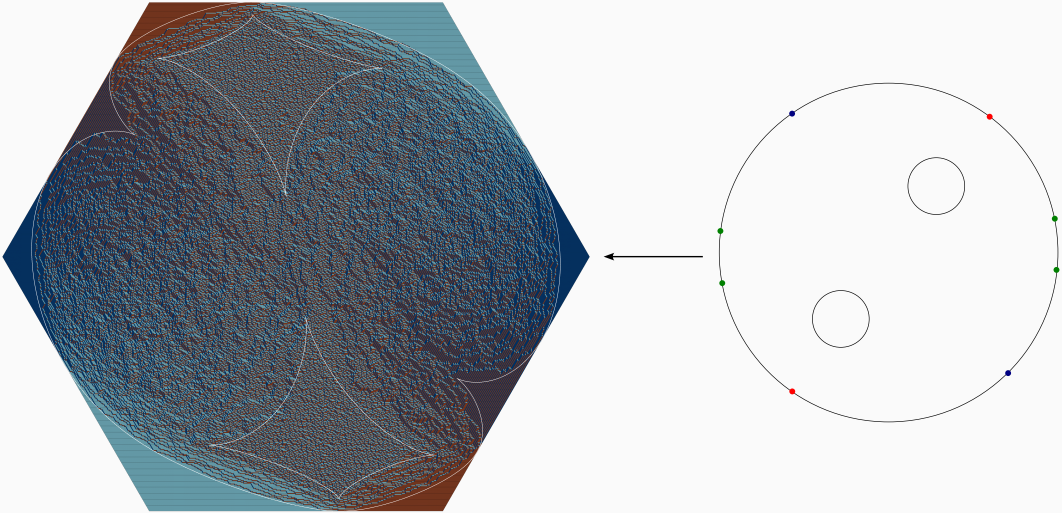


Weighted random Aztec diamond

Question

Hard: What does minimizer h look like?

Teaser: Inverse Problem



Setup

The Weighted Dimer Model

- Measure on dimer configurations D (perfect matchings):

$$\mathbb{P}(D) = \frac{1}{Z} \prod_{e \in D} \nu(e).$$

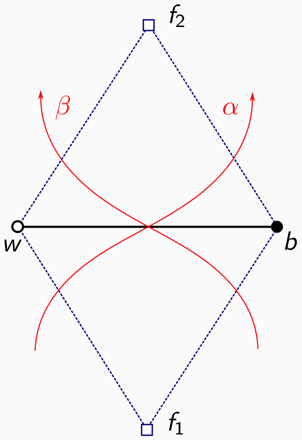
- Physical weights are face weights:

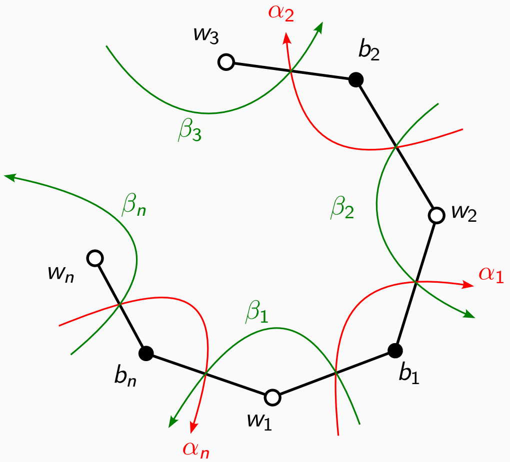
$$W_f = \frac{\nu(e_1)\nu(e_3)\dots\nu(e_{2n-1})}{\nu(e_2)\nu(e_4)\dots\nu(e_{2n})}.$$

- Kasteleyn condition:

$$\mathrm{sign}(W_f) = (-1)^{(n+1)}.$$

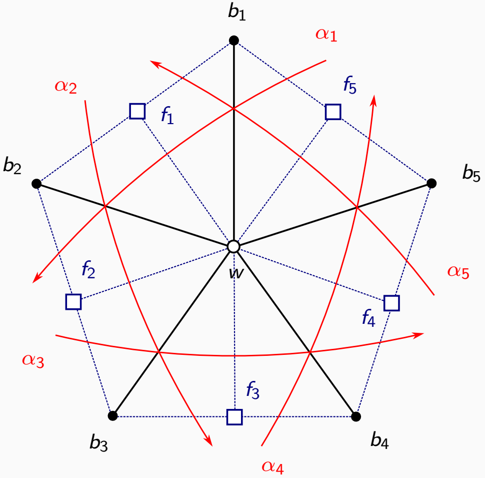
Quad Graph and Train Tracks





Fock weights

- **Data:** Compact Riemann surface $\mathcal{R}$, train track parameters α_i.



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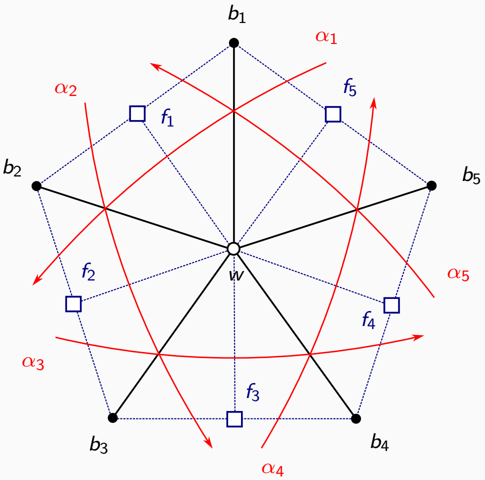
- ψ picks up a zero or a pole at $\alpha \in \mathcal{R}$ whenever crossing a train track.



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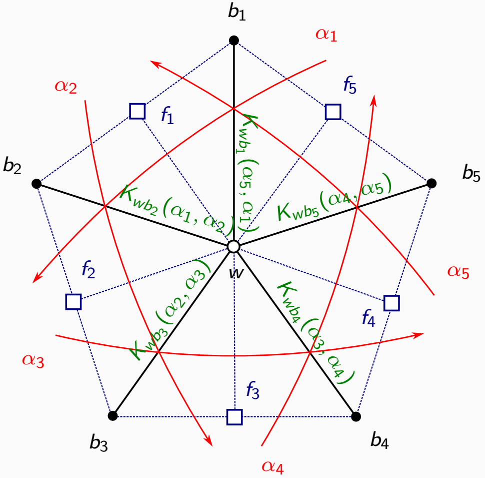
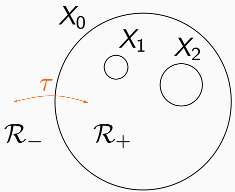
- ψ picks up a zero or a pole at $\alpha \in \mathcal{R}$ whenever crossing a train track.
- Fock weights [Fock '15]:

$$\sum_{k=1}^{n} K_{wb_k}(\alpha_{k-1}, \alpha_k)\psi_{b_k}(P) = 0,$$



Dimers from Fock weights

- $\mathcal{R}$ M-curve with antiholomorphic involution τ and fixed ovals $X_0, \ldots, X_g$.



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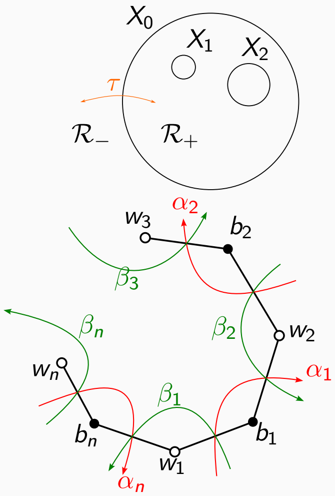

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- Ordering condition on α_i, β_i gives

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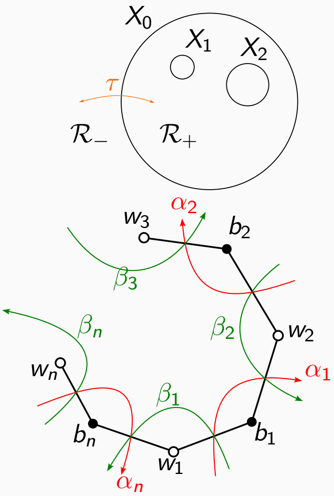
In general notion of minimal graphs [Boutillier, Cimasoni, de Tilière - '20, '21].



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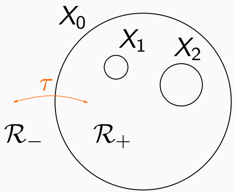
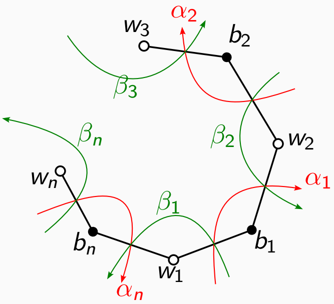
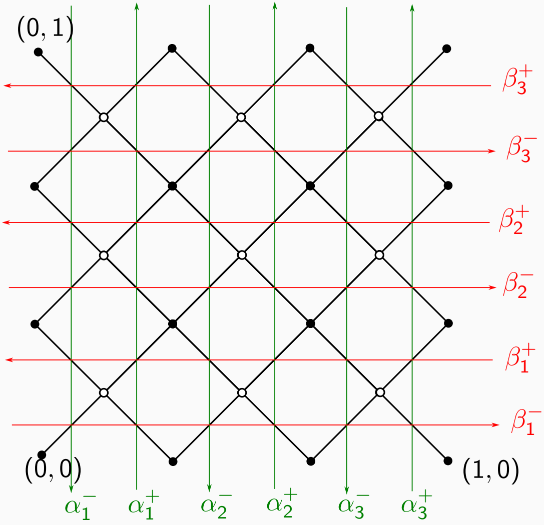
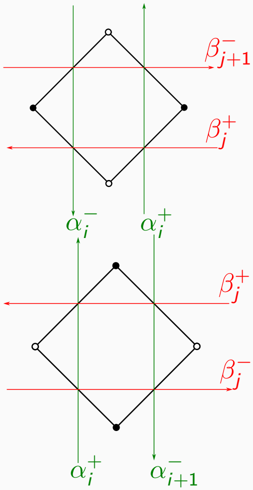
- For $g = 0$ these are isoradial weights. [Kenyon '02, Kenyon-Okounkov '03]





Quasi-periodic weights: square grid





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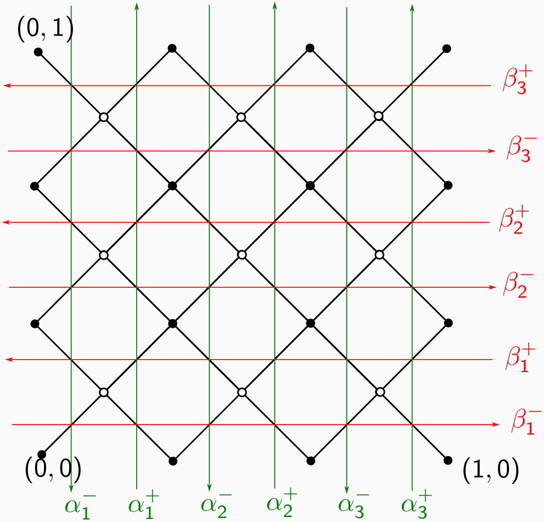
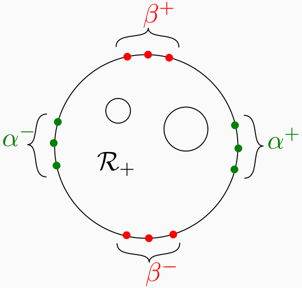
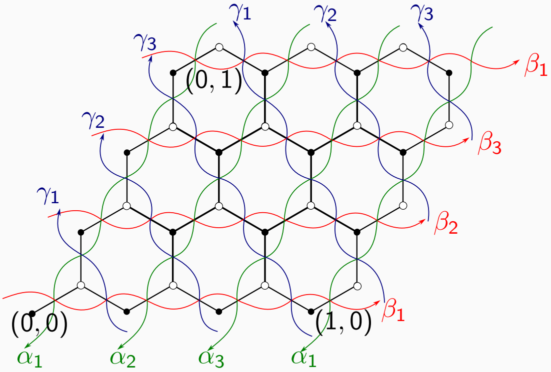


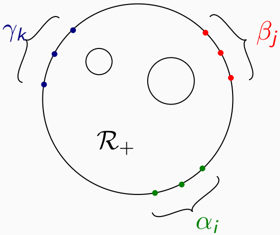
Kasteleyn

$\Updownarrow$



Quasi-periodic weights: hex grid





Quasi-periodic weights

$$W_f = \prod_{i=1}^{n} \frac{\theta[\Delta]\left(\int_{\alpha_i}^{\beta_i} \boldsymbol{\omega}\right)}{\theta[\Delta]\left(\int_{\beta_i}^{\alpha_{i+1}} \boldsymbol{\omega}\right)} \frac{\theta\left(\eta(f_{2i}) + D\right)}{\theta\left(\eta(f_{2i-1}) + D\right)}$$

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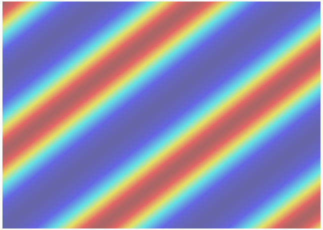
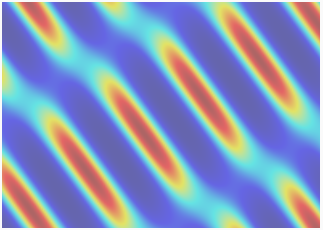
Train track parameters repeat periodically $\not\Rightarrow$ weights K_{wb}, W_f periodic.



(a) W_f for $g = 1$



(b) W_f for $g = 2$

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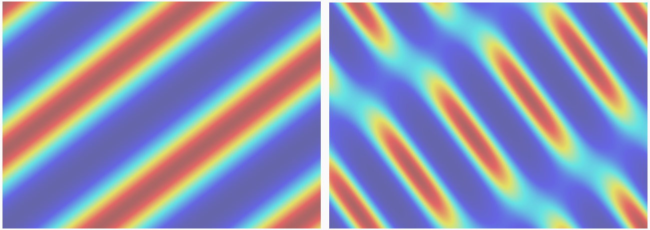

(a) W_f for $g = 1$ **(b)** W_f for $g = 2$

Weights periodic iff $\sum_i(\alpha_i^- - \alpha_i^+), \sum_i(\beta_i^- - \beta_i^+)$ principal divisors.

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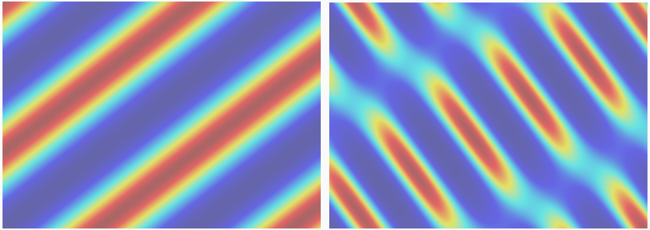

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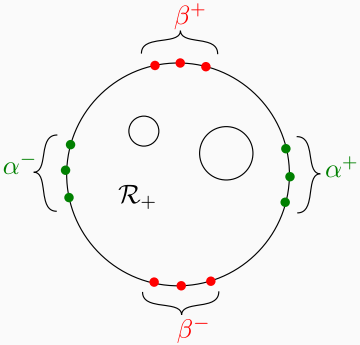
Covers all doubly periodic weights [Kenyon, Okounkov, Sheffield '07], $\mathcal{R}$ Harnack curve [Mikhalkin '00].

Algebro-geometric description

Amoebas, surface tension, etc.

- **Goal**: Describe limiting objects algebro-geometrically.
- **Data** $\mathcal{S}$: M-curve $\mathcal{R}$ with antiholomorphic involution τ. $\{\alpha_i^{\pm}, \beta_j^{\pm}\} \in X_0$ with clustering condition.



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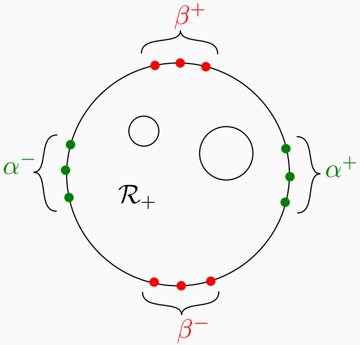
- $d\zeta_1, (d\zeta_2)$ normalized meromorphic differentials with residues ∓ 1 at $\alpha_i^{\pm}, (\beta_i^{\pm})$ and zero a periods.
- $\zeta_k(P) = \int^P d\zeta_k = x_k + iy_k$ well defined on $\mathcal{R}_+$.



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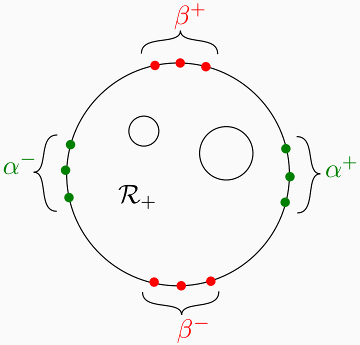
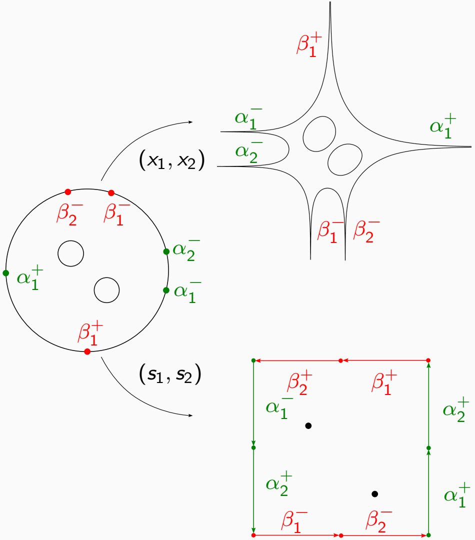
- **Proposition**: $(x_1, x_2), (y_1, y_2)$ coordinates of $\mathcal{R}_+^{\circ}$. [Krichever '14]



$(s_1, s_2) = \frac{1}{\pi}(y_2, -y_1)$.

Amoeba and polygon map

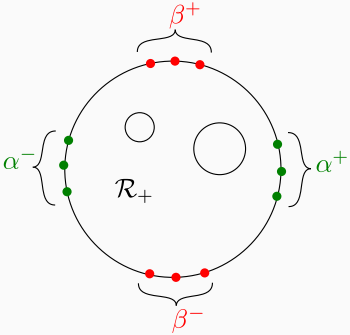


Amoebas, surface tension, etc.

- Ronkin function and surface tension

$$\rho(P) = \rho(x_1, x_2) = -\frac{1}{\pi}\,\mathrm{Im}\int^P \zeta_2 d\zeta_1 + x_2 s_2$$

$$\sigma(P) = \sigma(s_1, s_2) = \frac{1}{\pi}\,\mathrm{Im}\int^P \zeta_2 d\zeta_1 - x_1 s_1$$



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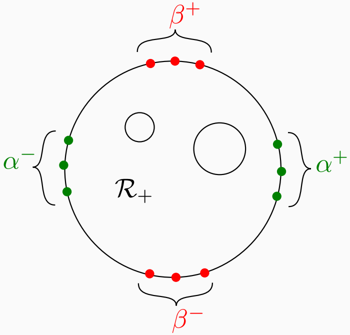
- Legendre dual

$$\nabla\sigma(s_1, s_2) = (x_1, x_2), \quad \nabla\rho(x_1, x_2) = (s_1, s_2).$$



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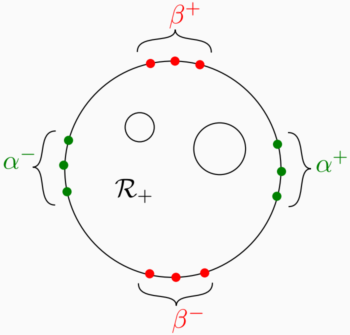
- Definitions agree with algebraic ones in doubly periodic case.



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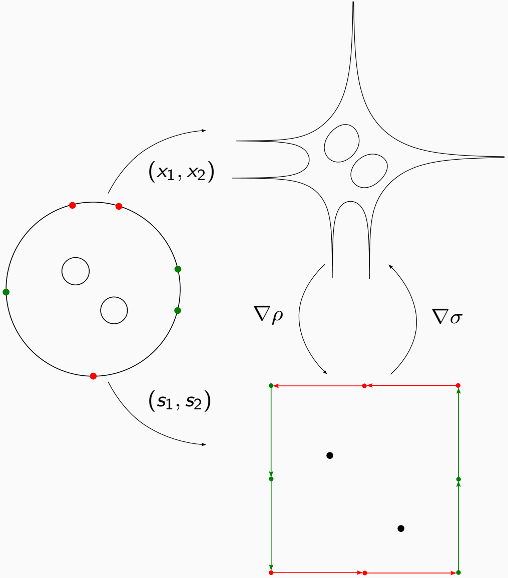
Ronkin function and Surface tension



Height function

- Doubly periodic: $h \to \arg\min_f \int_\Omega \sigma(\nabla f)$. [Kenyon-Okounkov-Sheffield '07, Kuchumov '17]

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- In [BB]: purely variational proof, more general boundary conditions.

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Res	α_i^-	β_i^-	α_i^+	β_i^+
$d\zeta_1$	1	0	-1	0
$d\zeta_2$	0	1	0	-1
$d\zeta_3$	1	-1	1	-1

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- $\#$ zeros $= \#$ poles $+ 2g - 2$

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- Define $d\zeta = d\zeta_{(u,v)} := -ud\zeta_2 + vd\zeta_1 + d\zeta_3$ [Berggren-Borodin '23, Kenyon-Prause '23] for Aztec diamond.
- $\#$ zeros $= \#$ poles $+ 2g - 2$
- For any $(u, v) \in (-1, 1)^2$ have:
  - 2 zeros on any inner oval
  - 1 zero between $\alpha_i^\pm, \alpha_{i+1}^\pm$, also $\beta_i^\pm, \beta_{i+1}^\pm$.
  - 2 free zeros.

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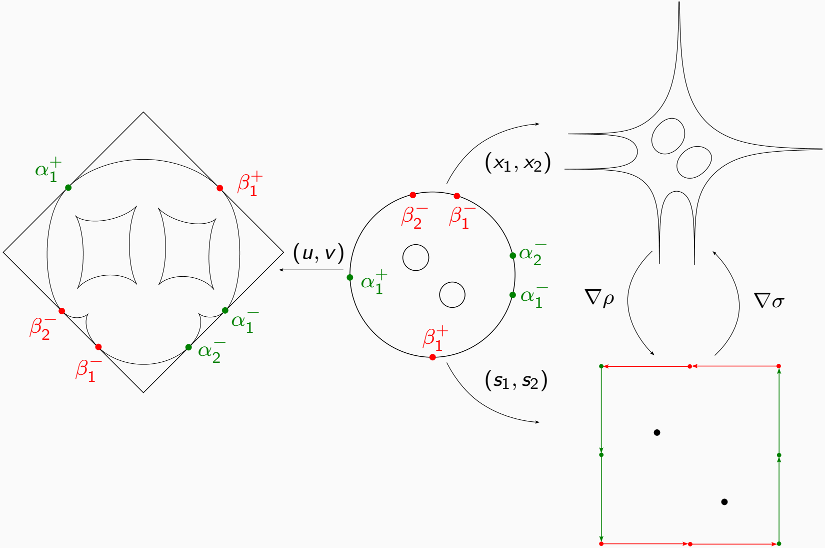
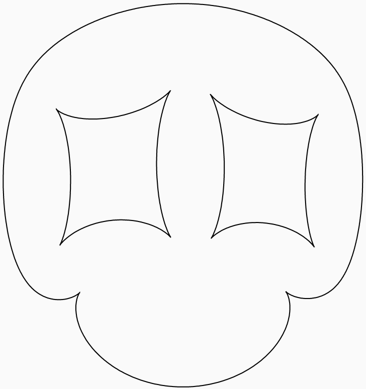
All Maps



α_1^+
β_1^+
β_2^-
α_1^-
α_2^-
β_1^-
(u, v)
β_2^-
β_1^-
α_2^-
α_1^+
α_1^-
β_1^+
(x_1, x_2)
(s_1, s_2)
$\nabla\rho$
$\nabla\sigma$

Geometric properties

- $\text{div}_{(u,v)}(x_1, x_2) = \text{div}_{(u,v)}(y_1, y_2) = 0.$



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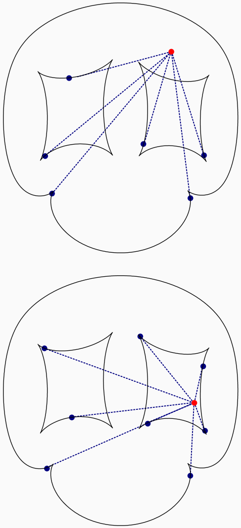
    - These are all (non-free) zeros.



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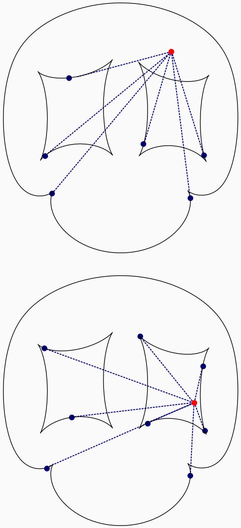

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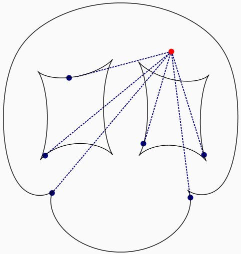
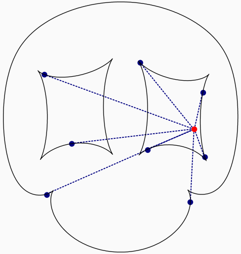
- Paralellity on real ovals: $(u', v')||(x_1', x_2')$.





The complex height function

- Define height function $H(u,v) = \int^P d\zeta_{(u,v)}$.
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- **Theorem** [BB]: g is magnetic tension minimizer:

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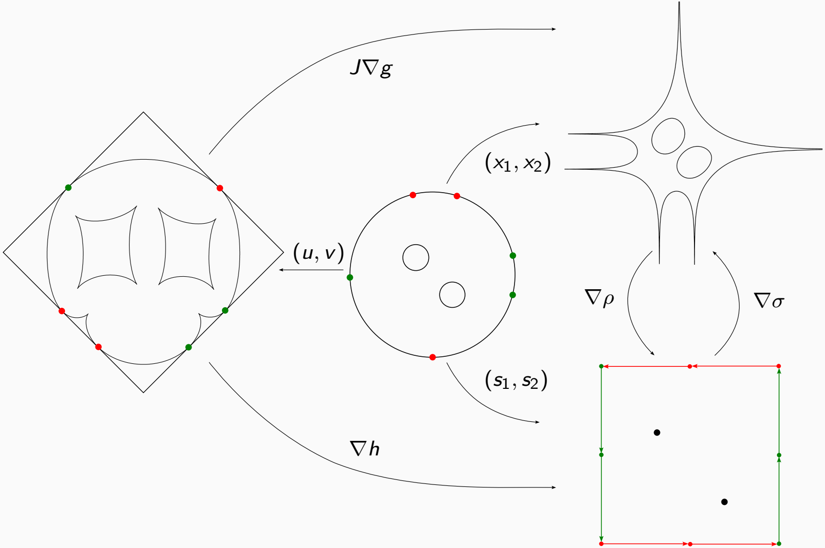
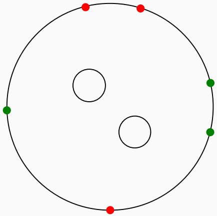
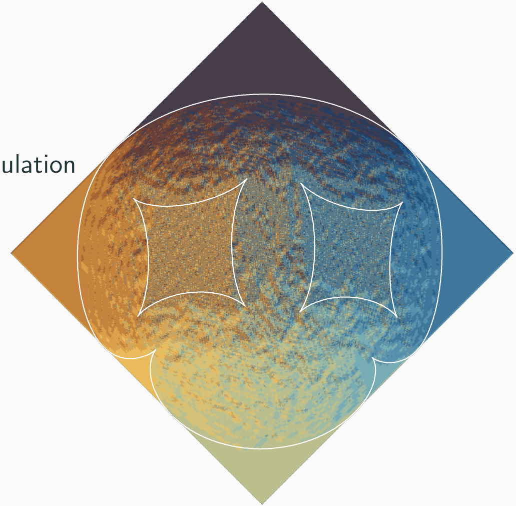


Computation

- All these formulas can be efficiently computed via Schottky uniformization.
- Pictures shown are actual computations, not just illustrations. github.com/nikolaibobenko/FockDimerSimulation
- Theoretical predictions match simulations on practical scales.





Proof idea

- Minimize $\int_{[-1,1]^2} \sigma(\nabla h)$.
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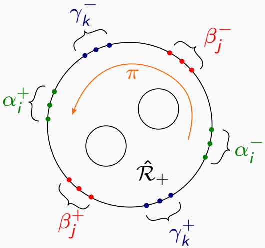
- Have non-differentiabilities in $\sigma \implies$ generalized Euler-Lagrange in terms of subgradients.
- $g = 0$: [Astala, Duse, Prause, Zhong, '20].
- In general follows from existence of extension of g to gas bubbles and frozen regions.

Hexagonal case

- $\hat{\mathcal{R}}$ double cover of $\mathcal{R}$ with symmetry $\hat{\mathcal{R}}/\pi = \mathcal{R}$.



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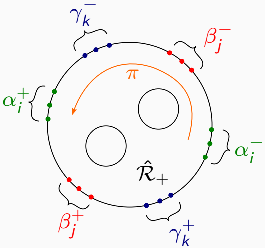

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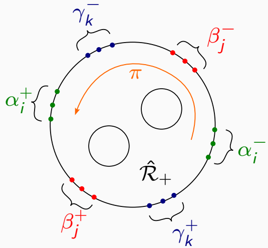

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- All same properties as before
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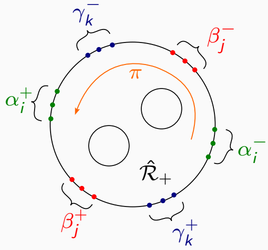

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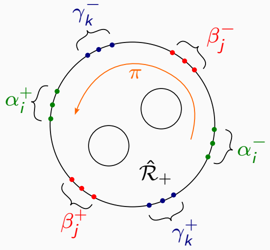
- **Theorem**: h is the surface tension minimizer for the hexagon.



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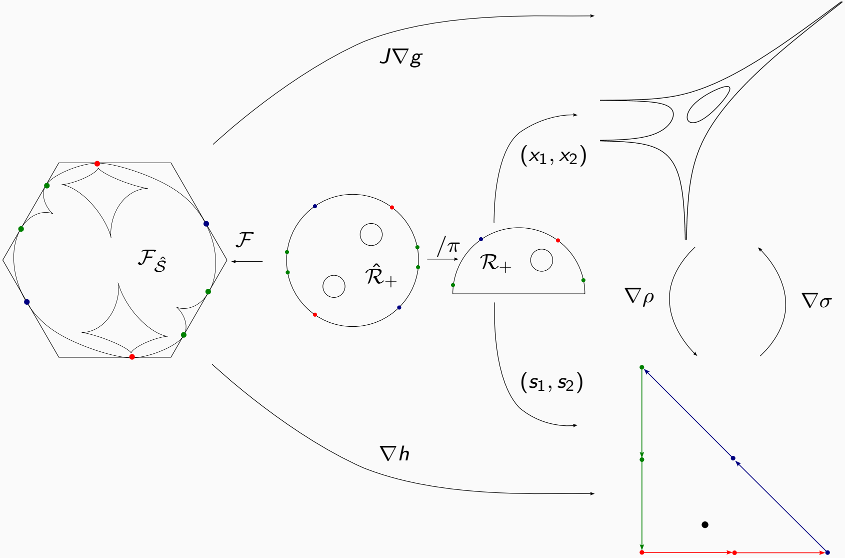
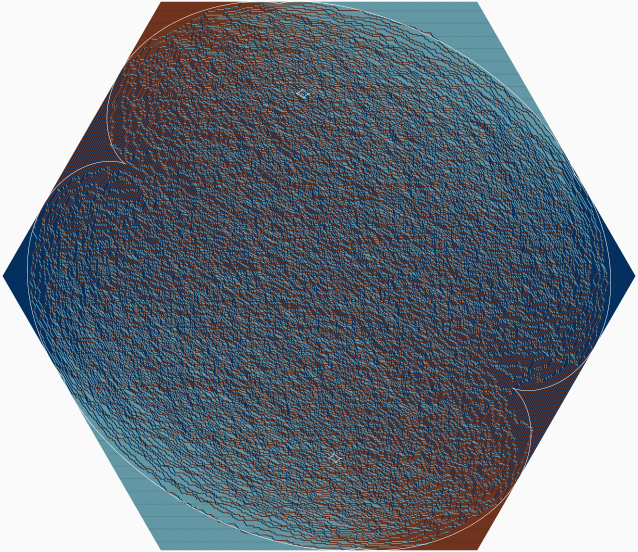
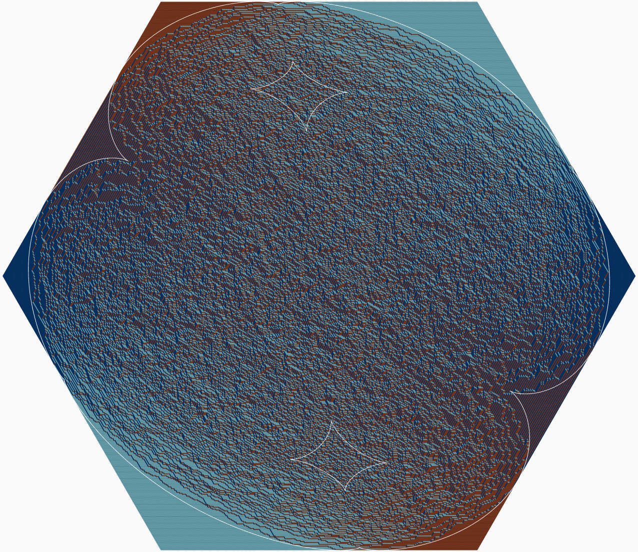


Gallery



Gallery



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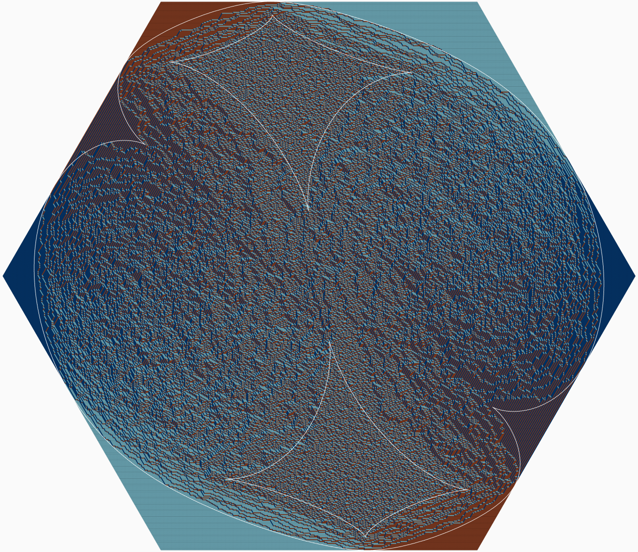

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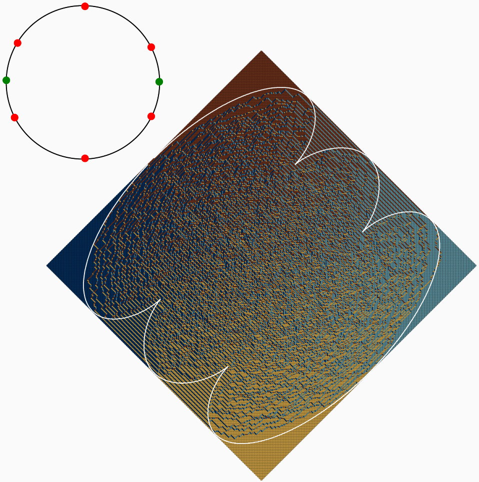

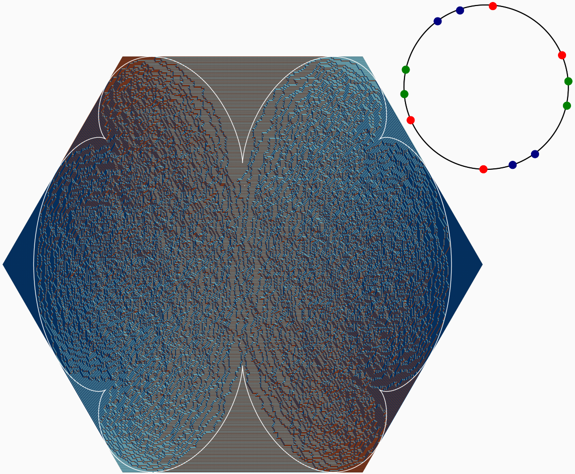

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